§6: Integration

<u>Obj</u>: We will study a process that in an same sense mirror to what we have done w/ derivatives.

<u>Warm up</u>: Compute $\int_0^1 (4x^3 + 3x^2)\,dx$.

$$\frac{d}{dx}(x^4 + x^3) = 4x^3 + 3x^2$$

$$\text{F.T.C} \Rightarrow \int_0^1 (4x^3 + 3x^2)\,dx = (x^4 + x^3)\Big|_{x=0}^{x=1}$$

$$= (1^4 + 1^3) - (0^4 + 0^3) = 2$$

<u>Def</u>: If $f(x)$ and $g(x)$ are functions satisfying $f'(x) = g(x)$, then $f(x)$ an "antiderivative" of $g(x)$.

<u>Ex</u>:
$$3x^2 = \frac{d}{dx}(x^3) = \frac{d}{dx}(x^3 + 2)$$

$$\frac{1}{2\sqrt{x}} = \frac{d}{dx}(\sqrt{x}) \qquad \frac{1}{x} = \frac{d}{dx}(\ln(x))$$

$$\sqrt{x} = x^{\frac{1}{2}}$$

<u>Rmk</u>: Suppose $f(x)$ and $g(x)$ are functions such that $f'(x) = g(x)$. Let $c \in (-\infty, \infty)$. Then

$$\frac{d}{dx}(f(x) + c) = \frac{d}{dx}(f(x)) + \cancel{\frac{d}{dx}(c)}^{\,0}$$

$$= \frac{d}{dx}\left(f(x)\right)$$

$$= g(x).$$

That is, $f(x)+c$ is also an antider. of $g(x)$.

<u>Def</u>: Let $f(x)$ be a function. Suppose all antider.'s of $f(x)$ are of the form $F(x)+c$ for $c \in (-\infty, \infty)$. We write the "indefinite integral" of $f(x)$ by

$$\int f(x)\,dx = F(x)+c$$

* $\int_a^b f\,dx$ = number vs. $\int f(x)$ is a family of functions *

<u>Rmk</u>:

1. $\int_a^b f(x)\,dx$ is a number

2. $\int f(x)dx$ is a family of functions

Ex: Find $\int 8x^3\,dx$ and $-\int \frac{1}{x^2}\,dx$.

$$\int 8x^3\,dx = 8\cdot\int x^3\,dx = 2\cdot 4\cdot\int x^3\,dx = 2\cdot\int 4x^3\,dx$$

$$= 2\cdot x^4 + C$$

$$-\int \frac{1}{x^2}\,dx = \int\left(-\frac{1}{x^2}\right)dx = \int\left(-x^{-2}\right)dx = x^{-1} + C$$

Ex: Show that $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$.

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) + \overset{0}{\cancel{\frac{d}{dx}(c)}} = \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{n+1}\cdot x^{n+1}\right) = \frac{1}{n+1}\,\frac{d}{dx}\left(x^{n+1}\right)$$

$$= \frac{\overset{1}{\cancel{n+1}}}{\cancel{n+1}}\,x^{n+1-1} = x^n$$

Ex: Show that $\int k\,dx = kx + C$ for $k \in (-\infty, \infty)$.

$$\frac{d}{dx}(kx + c) = \frac{d}{dx}(kx) + \overset{0}{\cancel{\frac{d}{dx}(c)}}$$

$$= k\cdot\frac{d}{dx}(x)$$

$$= k\cdot 1$$

$$= k$$

$$\left[\; x^1 = x \qquad \frac{d}{dx}(x) = 1\cdot x^{1-1} = 1\cdot x^0 = 1\cdot 1 = 1 \;\right]$$

Ex: Find $\int (3x + x^2)\,dx$

$$3x = 3 \cdot \frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{d}{dx}\left(\frac{3}{2}x^2\right)$$

$$x^2 = \frac{d}{dx}\left(\frac{1}{3}x^3\right)$$

$$\frac{d}{dx}\left(\frac{3}{2}x^2 + \frac{1}{3}x^3\right) = 3x + x^2 \implies \int (3x + x^2)\,dx = \frac{3}{2}x^2 + \frac{1}{3}x^3 + C$$

Ex: Find the following:

1. $\int x^5\,dx$
2. $\int t^8\,dt$
3. $\int 12x^3\,dx$
4. $\int (q^3 - 6q^2)\,dq$

1. $\int x^5\,dx = \frac{1}{6}x^6 + C$

$$x^5 = \frac{d}{dx}\left(\frac{1}{6}x^6\right)$$

2. $\int t^8\,dt = \frac{1}{9}t^9 + C$

3. $\int 12x^3\,dx = 3x^4 + C$

$$12x^3 = 12 \cdot \frac{d}{dx}\left(\frac{1}{4}x^4\right) = \frac{d}{dx}\left(\frac{12}{4}x^4\right) = \frac{d}{dx}(3x^4)$$

4. $\int (q^3 - 6q^2)\,dq = \frac{1}{4}q^4 - 2q^3 + C$

$$q^3 = \frac{d}{dq}\left(\frac{1}{4}q^4\right)$$

$$-6q^2 = \frac{d}{dq}\left(-6 \cdot \frac{1}{3}q^3\right) = \frac{d}{dq}\left(-2q^3\right)$$

$$\frac{d}{dq}\left(\frac{1}{4}q^4 - 2q^3\right) = q^3 - 6q^2$$

Ex: Show $\int \frac{1}{x}\,dx = \ln|x| + c.$

absolute value

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ for $x > 0$

$\Rightarrow \int \frac{1}{x}\,dx = \ln(x) + C$ for $x > 0$

$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x}$ for $x < 0$

$\Rightarrow \int \frac{1}{x}\,dx = \ln(-x) + C$ for $x < 0$

Thus, $\int \frac{1}{x}\,dx = \ln|x| + C$ for $x \in (-\infty, \infty)$.

Ex: Show $\int e^{kx}\,dx = \frac{1}{k} e^{kx} + C$ for $k \in (-\infty, \infty)$ a non zero constant.

$\frac{d}{dx}(e^{kx}) = e^{kx} \cdot \frac{d}{dx}(kx) = k \cdot e^{kx} \Rightarrow$

$\frac{1}{k}\frac{d}{dx}(e^{kx}) = e^{kx} \Rightarrow$

$\frac{d}{dx}\left(\frac{1}{k} e^{kx}\right) = e^{kx} \Rightarrow \int e^{kx}\,dx = \frac{1}{k} e^{kx} + C$